Ringed spaces (See Har II.2, Shaf V.3.1)

To each ring R, we now have associated a topological space $\operatorname{Spec} R$ and a sheaf of rings $\mathcal{O}$. We want to show that this is functorial, so we first need a category in which $(\operatorname{Spec} R, \mathcal{O})$ is an object.

Def: A <u>ringed space</u> is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X.

Ex: $(\operatorname{Spec} R, \mathcal{O})$ is a ringed space

Ex: A topological space X together w/ the sheaf of continuous $\mathbb{R}$-valued functions is a ringed space. (i.e. not all ringed spaces look like $\operatorname{Spec} R$).

Def: A <u>morphism of ringed spaces</u> from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ of a continuous map $f: X \longrightarrow Y$ and a map $f^\# : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$ of sheaves of rings. (i.e. over each open $U \subseteq Y$ a ring homomorphism $\mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$ that commutes with restriction maps.)

An important property of $(\operatorname{Spec} R, \mathcal{O})$ is the fact that every stalk $\mathcal{O}_p$ is a local ring, which is reflected in the following:

Def: A ringed space $(X, \mathcal{O}_X)$ is a <u>locally ringed space</u> if for each $P \in X$, the stalk $\mathcal{O}_{X,P}$ is a local ring.

We want morphisms of loc. ringed spaces to respect the local ring structure on the stalks. That is:

Def: If A and B are local rings w/ max'l ideals m_A and m_B respectively, then a homomorphism $\varphi : A \longrightarrow B$ is a <u>local homomorphism</u> if $\varphi^{-1}(m_B) = m_A$, or equivalently $\varphi(m_A) \subseteq m_B$.

We know that a morphism $(f, f^\#)$ of ringed spaces induces maps between the stalks of $\mathcal{O}_Y$ and $f_*\mathcal{O}_X$, but we want maps $\mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$.

Let $P \in X$. Then by definition we have

$$\mathcal{O}_{Y,f(P)} = \varinjlim_{U \ni f(P)} \mathcal{O}_Y(U) \xrightarrow{f^\#} \varinjlim_{U \ni f(P)} \mathcal{O}_X(f^{-1}U) \longrightarrow \varinjlim_{V \ni P} \mathcal{O}_X(V) = \mathcal{O}_{X,P}$$

$$(s, f^{-1}U) \longmapsto (s, f^{-1}U)$$

Call this composition $f^\#_p$.

This is well-defined since the $f^{-1}(U)$ s.t. $U \ni f(P)$ are a subset of all the open sets containing P. So if $(s, f^{-1}(U)) \sim (s', f^{-1}(U'))$ in the first direct limit then the will also be equivalent in the second.

Def: If X and Y are locally ringed spaces, then a morphism $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces is a <u>morphism of locally ringed spaces</u> if for each $P \in X$, $f^{\#}_P : \mathcal{O}_{Y, f(P)} \to \mathcal{O}_{X,P}$ is a local homomorphism. (It's an <u>isomorphism</u> if f and $f^{\#}$ are.)

$(\operatorname{Spec} R, \mathcal{O})$ is clearly a locally ringed space, but we need to show that ring homomorphisms induce morphisms of locally ringed spaces.

Theorem: If $\varphi : R \to S$ is a morphism of rings, then φ induces a natural morphism of locally ringed spaces

$$(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y),$$

where $X = \operatorname{Spec} S$, $Y = \operatorname{Spec} R$.

Pf: $f : X \to Y$ is the map we've already defined:
$f(P) = \varphi^{-1}(P)$.

We first define $f^{\#}$ on distinguished open sets by

$$
\begin{array}{ccc}
f^\# : \mathcal{O}_Y(D(a)) & \longrightarrow & f_* \mathcal{O}_X(D(a)) \\
\parallel & & \parallel \\
R_a & & \mathcal{O}_X(f^{-1}(D(a))) \\
 & & \parallel \\
 & & \mathcal{O}_X(D(\varphi(a))) \\
 & & \parallel \\
 & & S_{\varphi(a)}
\end{array}
$$

where $\frac{r}{a} \longmapsto \frac{\varphi(r)}{\varphi(a)}$.

(Note that if $\varphi(a) = 0$, then $S_{\varphi(a)} = 0$ and this is the zero map.)

This uniquely extends to a morphism on each open $U \subseteq Y$

If $P \in X = \operatorname{Spec} Y$, then $f(P) = \varphi^{-1}(P) \in \operatorname{Spec} X$, and the induced map on stalks is

$$R_{\varphi^{-1}(P)} \longrightarrow S_P,$$

which is local by construction. $\square$

Conversely, all morphisms $\operatorname{Spec} B \to \operatorname{Spec} A$ arise uniquely in this way. That is:

Theorem: If R and S are rings then any morphism of locally ringed spaces $f: \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is induced (uniquely) by a homomorphism $\varphi: R \longrightarrow S$.

Thus, there is a one-to-one correspondence between such morphisms.

Pf: There is only one possible candidate for φ (hence uniqueness): the induced map on global sections. So set φ to be

$$f^{\#}: \underset{\substack{\parallel \\ R}}{\Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})} \longrightarrow \underset{\substack{\parallel \\ S}}{\Gamma(\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S})}$$

Thus, we just need to check that $(f, f^{\#})$ is the map induced by φ.

We know that φ is compatible with the map on stalks, so

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & S \arrow[d] \\ R_{f(P)} \arrow[r, "f_P^{\#}"] & S_P \end{tikzcd}$$

commutes. But $f_P^{\#}$ is local, which means that $(f_P^{\#})^{-1}(P) = f(P)$. Commutativity of the diagram implies that $\varphi^{-1}(P) = f(P)$.

So the map f is the one induced by φ.

Now maps $f^{\#}: R_a \longrightarrow S_{\varphi(a)}$ over $D(a)$

are also compatible w/ φ, so they must be those induced by φ, so $f^{\#}$ is induced by φ, and we're done. $\square$

Remark: In general, if $\psi: \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, and we know $\psi(U)$ for each element U of a basis, then we can recover ψ. (Use sheaf condition.)